Math Review

Economics 100A

Fall 2021

1 Functions

1.1 Univariate Functions

In upper-division economics, you will be expected to be comfortable with graphing linear functions, solving for their slopes, and plotting their intercepts. Now, we are going to quickly review how to solve for slopes and intercepts of univariate (single variable) functions. Recall that the general form for a linear graph is $y = mx + b$ where m is the slope and b is the intercept.

Function	Slope	y-intercept	x-intercept
$y = 2x$	2	0	0
$y = -2x + 4$	-2	4	2
$x + y = 5$	-1	5	5
$3x + 4y = 10$	$-3/4$	10/4	10/3
$y/2 = 3 - x$	-2	6	3

These functions are all pretty straightforward, so be sure you can solve for their properties. As well, make sure you can graph nonlinear functions. Some popular examples that show up in this class include:

1. $y = \sqrt{x}$
2. $y = x^2$
3. $y = ln(x)$

This is not an exhaustive list, but you should definitely familiarize yourself with the shapes of these functions. It will come in handy later!

1.2 Multi-variate Functions

Since Math 10/20C is a prerequisite for this course, you are expected to have familiarity with multivariate calculus and functions. In this class, we will mainly work with functions that take on the form $z = f(x, y)$. In reality, you will not be expected to draw the three-dimensional graphs. However, just for reference, we can graph $z = \sqrt{xy}$. This graph would look something figure one.

Drawing these 3D graphs is incredibly difficult without graphing software, so we will take a different approach. Instead of graphing functions like this, we will create level curves for each function. This

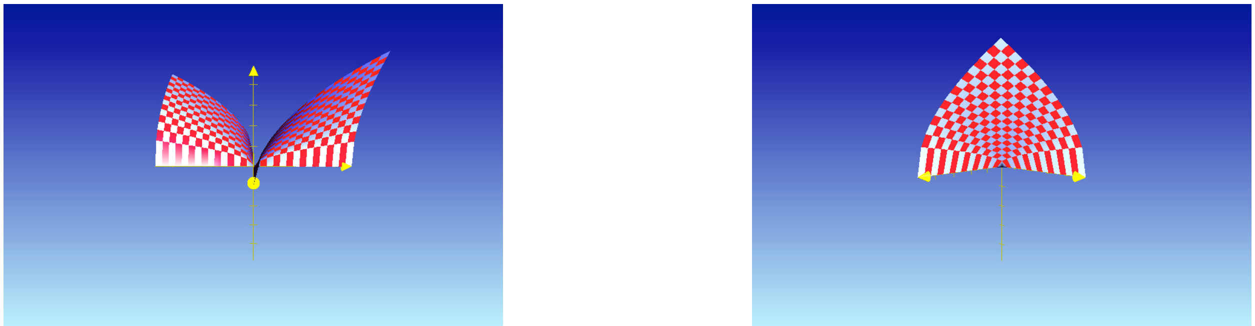

Figure 1: $z = \sqrt{xy}$

is what cartographers do to represent different three-dimensional terrain. Imagine taking our three-dimensional graph and slicing through a point, $z*$. This is how we get a single "level." I have added an image which shows how this slice works at $z = 10$. We can in fact do this for all levels across z, giving us the figure to the right.

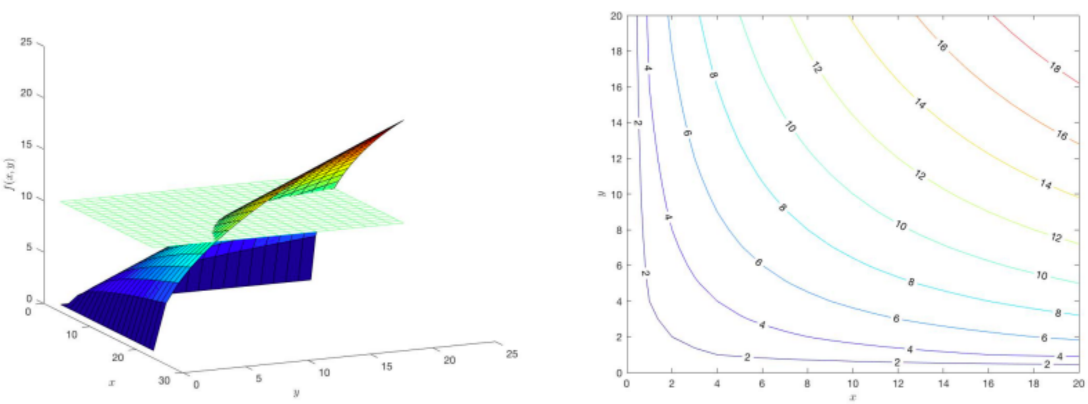

Figure 2: Level curve at $z = 10$.

2 Derivatives

2.1 Basic Rules

You should be comfortable with single-variable derivatives and their interpretations. Throughout this class, we will be differentiating multivariate functions. In case you need a brief reminder, the derivative of a function tells us the instantaneous rate of change, or the slope at a single point. Most derivatives we do in this class can be done using the power rule, but in some cases you will need to use the chain rule. In any event, we can go over what those look like.

2.1.1 Rules

1. If $f(x) = x^\alpha$ then $f'(x) = \alpha x^{\alpha-1}$. This only applies when α is constant.
2. $\dfrac{d}{dx} ln(x) = \dfrac{1}{x}$.
3. $\dfrac{d}{dx} ln[f(x)] = \dfrac{1}{f(x)} f'(x)$
4. $\dfrac{d}{dx} c = 0$, where c is a constant.
5. $\dfrac{d}{dx}\ c \cdot f(x) = c \cdot f'(x)$
6. If $f(x) = u(x) + v(x)$, then $f'(x) = v'(x) + u'(x)$
7. If $f(x) = u(v(x))$, then $f'(x) = \dfrac{du(v)}{dv} \cdot \dfrac{dv(x)}{dx} = u'(v(x)) \cdot v'(x)$
8. If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$
9. If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{v(x)^2}$

2.2 Partial Derivatives

Partial derivatives are derivatives of multivariate functions with respect to a single variable. We treat each other variable as a constant.

Examples

Function	Derivative of x	Derivative of y
$f(x) = x^2$	$2x$	0
$f(x) = x$	1	0
$f(x, y) = xy$	y	x
$f(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$	$\frac{1}{3} x^{\frac{-2}{3}} y^{\frac{2}{3}}$	$\frac{2}{3} y^{\frac{-1}{3}} x^{\frac{1}{3}}$
$f(x, y) = \log(x^{\frac{1}{3}} y^{\frac{2}{3}})$	$\frac{1}{3x}$	$\frac{2}{3y}$
$f(x, y) = \log(x^{\frac{1}{3}} y^{\frac{2}{3}}) + e^{3x}$	$\frac{1}{3x} + 3e^{3x}$	$\frac{2}{3y}$
$f(x, y) = \log(x) + y$	$\frac{1}{x}$	1

There is no royal road to math. If you feel uncomfortable with any of these derivatives, review and practice. Practice is the only way you actually improve in math; it is not a spectator sport.

3 Optimization

3.1 Constrained Optimization

Economics is concerned with efficiency: how do we efficiently allocate our scarce goods? Think about your own personal experiences going to the store. You typically have some budget which allows you to purchase a certain number of goods. Following your budget constraint, you buy whatever bundle of goods satisfies your needs and/or desires. In economic terms, you are maximizing utility, subject to some budget constraint. We will go into what exactly that means in a few sections, but for now just think about maximizing happiness subject to a certain constraint: money, time, and so on.

We need to identify three things:

1. *Objective Function*: What function are we maximizing?
2. *Constraint*: What are we limited to?
3. *Choice Variables*: What variables will we maximize?

In terms of math, we write

$$max f(x, y)$$
$$\text{s.t.} \quad h(x, y) \geq 0$$

What we are doing here is maximizing (though we could minimize, but we would have to write $\min(f(x, y))$ some function subject to a constraint. In this case, our objective function is $f(x, y)$ and our constraint is $h(x, y)$. Our choice variables are x and y in this case, and in most cases for this class.

If you are interested in the math behind the constraint: we are restricting the number of sets that are feasible. We write this as an inequality. However, in our class, we will typically write our constraint as an equality. This is more for economic reasons, as you will soon see.

3.1.1 Substitution Method

1. Rewrite the constraint in terms of x or y.
2. Substitute the constraint into the original function for either x or y.
3. Take the first order condition with respect to the single variable in the objective function.
4. Rearrange and solve for the choice variable. Then, use the constraint to solve for the other choice variable.

Example Say we have $f(x, y) = xy$ s.t. $x + y = 10$. Step one tells us to rewrite the constraint in terms of a single variable, so we have

$$y = 10 - x$$
$$f(x,y) = x(10-x)$$
$$f(x,y) = 10x - x^2$$
$$\frac{d}{dx}(10x - x^2) = 10 - 2x$$
$$2x = 10$$
$$x = 5$$

Substitute our new x into the constraint to get $y = 10 - 5$ or $y = 5$.

Example Two Let's make things just a tad more difficult. Say we have to maximize x^2y s.t. $c - x - 2y = 0$. In this case we are not given a number for c, so we will solve for a general case.

$$y = \frac{c}{2} - \frac{x}{2}$$

Substitute the constraint into our objective function: $x^2(\frac{c}{2} - \frac{x}{2}) = \frac{cx^2}{2} - \frac{x^3}{2}$

Take the first order condition: $\frac{d}{dx} = cx - \frac{3}{2}x^2 = 0$

We can factor out an x to get $x(c - \frac{3x}{2})$

As you can see, x will either be 0 or $\frac{2c}{3}$. In this case, does 0 make sense for maximization? No, so the answer is $\frac{2c}{3}$. If we plug in to solve for y, we get: $\frac{c}{2} - \frac{c}{3} = \frac{c}{6}$.

Et voila. There we have our constrained optimization.

3.1.2 Lagrangian

The Lagrange multiplier method for solving constrained optimization is very powerful but more time consuming than the substitution method. Most students prefer using the substitution method, but it is good to be familiar with both. In any event, these notes cover how you would use a Lagrange multiplier.

1. Set up $\mathcal{L} = Objective + \lambda[constraint] = f(x,y) + \lambda[h(x,y)]$
   (a) Set your constraint equal to zero so that you have $c - x - y = 0$ inside the constraint
2. Take three first order conditions: one with respect to x, one with respect to y, and one with respect to λ.
3. Solve $\frac{df}{dx}$ and $\frac{df}{dy}$ in terms of λ and then equate the two.
4. Rewrite the above equation in terms of either x or y.
5. Substitute x or y into the budget constraint (i.e. the partial with respect to λ).

6. Finally, solve for x or y and use the budget constraint to solve for the other choice variable.

Example Let's go back to x^2y s.t. $c - x - 2y = 0$. Set up the Lagrange as follows: $\mathcal{L} = x^2y + \lambda[c - x - 2y]$

$$\frac{d\mathcal{L}}{dx} = 2xy - \lambda = 0.$$

$$\frac{d\mathcal{L}}{dy} = x^2 - 2\lambda = 0.$$

$$\frac{d\mathcal{L}}{d\lambda} = c - x - 2y = 0.$$

Rewrite both x and y in terms of λ and equate them.

$$\tfrac{1}{2}x^2 = 2xy$$

$$x = 4y$$

Substitute into the constraint

$$c - 4y - 2y = 0$$

$$c - 6y = 0$$

$$y = \tfrac{1}{6}c$$

Now that we have y, we can solve for x.

$$c - x - 2y = 0$$

$$c - x - 2(\tfrac{1}{6}c) = 0$$

$$x = c - \tfrac{2}{6}c$$

$$x = \tfrac{4}{6}c$$